Trinity College

Semester Two Examination, 2017

Question/Answer booklet

MATHEMATICS APPLICATIONS UNITS 3 AND 4

Section One: Calculator-free

SOLUTIONS

Student Number: In figures | | | | | | | | |

In words _____

Your name _____

Time allowed for this section

Reading time before commencing work: five minutes
Working time: fifty minutes

Materials required/recommended for this section

To be provided by the supervisor

This Question/Answer booklet
Formula sheet

To be provided by the candidate

Standard items: pens (blue/black preferred), pencils (including coloured), sharpener, correction fluid/tape, eraser, ruler, highlighters

Special items: nil

Important note to candidates

No other items may be taken into the examination room. It is **your** responsibility to ensure that you do not have any unauthorised material. If you have any unauthorised material with you, hand it to the supervisor **before** reading any further.

Structure of this paper

Section	Number of questions available	Number of questions to be answered	Working time (minutes)	Marks available	Percentage of examination
Section One: Calculator-free	7	7	50	52	35
Section Two: Calculator-assumed	12	12	100	98	65
				Total	100

Instructions to candidates

1. The rules for the conduct of examinations are detailed in the school handbook. Sitting this examination implies that you agree to abide by these rules.

2. Write your answers in this Question/Answer booklet.

3. You must be careful to confine your response to the specific question asked and to follow any instructions that are specified to a particular question.

4. Additional working space pages at the end of this Question/Answer booklet are for planning or continuing an answer. If you use these pages, indicate at the original answer, the page number it is planned/continued on and write the question number being planned/continued on the additional working space page.

5. Show all your working clearly. Your working should be in sufficient detail to allow your answers to be checked readily and for marks to be awarded for reasoning. Incorrect answers given without supporting reasoning cannot be allocated any marks. For any question or part question worth more than two marks, valid working or justification is required to receive full marks. If you repeat any question, ensure that you cancel the answer you do not wish to have marked.

6. It is recommended that you do not use pencil, except in diagrams.

7. The Formula sheet is not to be handed in with your Question/Answer booklet.

See next page

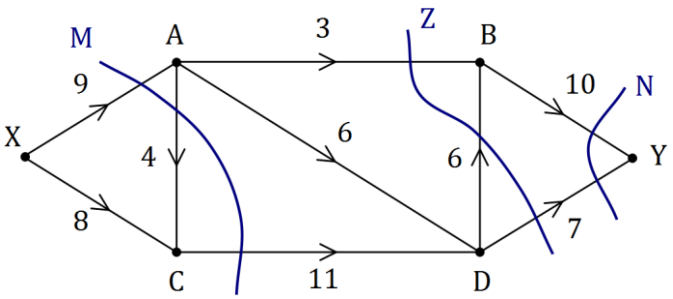
Section One: Calculator-free **35% (52 Marks)**

This section has **seven (7)** questions. Answer **all** questions. Write your answers in the spaces provided.

Working time: 50 minutes.

Question 1 **(6 marks)**

The network shows a system of pipes with the maximum capacity for each pipe, in litres per second, shown on the edges.



(a) Cut M passes through edges XA, AC and CD, and cut N passes through edges BY and DY. Show these cuts on the network and state their capacities. (2 marks)

Solution
See network Cut M has capacity 24 L/s and cut N has capacity 17 L/s.
Specific behaviours
✓ shows cut M and states capacity ✓ shows cut N and states capacity

(b) Determine the maximum flow through the system from X to Y by listing each path used and the flow along each path. (3 marks)

Solution (A)
$XABY = 3$ $XADBY = 6$ $XCDY = 7$ $\text{Total} = 16\ \text{L/s}$
Specific behaviours
✓ at least two paths with correct flow contribution ✓ all paths with correct flow contribution ✓ states maximum flow

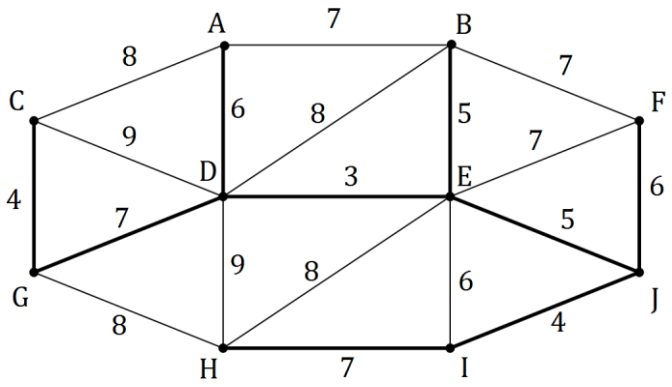
(c) Show cut Z on the network that has capacity equal to the maximum flow. (1 mark)

Solution
See diagram
Specific behaviours
✓ clearly shows cut

See next page

Question 2 **(5 marks)**

Cabling between ten distribution boards in a factory is to be upgraded to ensure the supply of electricity between all boards in an emergency. The upgrade costs between adjacent boards, in thousands of dollars, are shown on the edges in the weighted graph.



(a) Determine the minimum spanning tree for the graph, clearly showing it on the graph. (3 marks)

Solution
See diagram
Specific behaviours
✓ tree ✓ at least 7 correct edges ✓ all correct edges

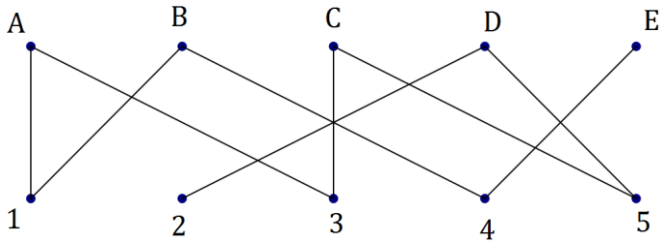
(b) Calculate the cost of upgrading the cabling that forms the minimum spanning tree. (2 marks)

Solution
Sum of edges =47 Upgrade cost is \$47 000.
Specific behaviours
✓ correct sum of edges ✓ correct units

See next page

Question 3 **(7 marks)**

Five people, A, B, C, D and E are to be allocated to five tasks, $1, 2, 3, 4$ and 5. The bipartite graph below shows the tasks that each of the five people can carry out.



(a) Explain why the graph is connected. (1 mark)

Solution
A path can be found to connect any vertex to all other vertices in the graph.
Specific behaviours
✓ clear explanation

(b) Explain why the graph is not a complete bipartite graph, and state the number of edges the graph would have if it was a complete bipartite graph. (2 marks)

Solution
All vertices from top set are not connected to all vertices in bottom set. $E = 5\times5 = 25$ edges
Specific behaviours
✓ clear explanation ✓ correct number of edges

(c) If person A is assigned to task 1, explain why a complete matching of people to tasks is not possible. (2 marks)

Solution
If A does task 1, then only task possible for B is 4 - but this is also the only task that E can do, and so one of them cannot do a task.
Specific behaviours
✓ uses task 4 ✓ clear explanation

(d) Determine a complete matching of people to tasks. (2 marks)

Solution
$A-3,\quad B-1,\quad C-5,\quad D-2,\quad E-4.$
Specific behaviours
✓ links E to 4 and D to 2 ✓ correct matching

See next page

Question 4 **(8 marks)**

(a) A sequence has the recursive definition $T_{n+1} = 0.5T_n - 10,\ T_1 = 64$. Determine the value of the first term of the sequence that is less than 0. (3 marks)

Solution
$T_2 = 0.5(64) - 10 = 22$ $T_3 = 0.5(22) - 10 = 1$ $T_4 = 0.5(1) - 10 = -9.5$ Value of term is -9.5
Specific behaviours
✓ calculates second term of sequence ✓ calculates next two terms ✓ states value of T_4

(b) Consider the arithmetic sequence $11, 14, 17, 20, \ldots$.

(i) Determine the 201^{st} term of the sequence. (3 marks)

Solution
$T_n = 11 + 3(n-1)$ $T_{201} = 11 + 3(201-1)$ $T_{201} = 611$
Specific behaviours
✓ uses form of n^{th} term rule ✓ substitutes 201 into rule ✓ determines T_{201}

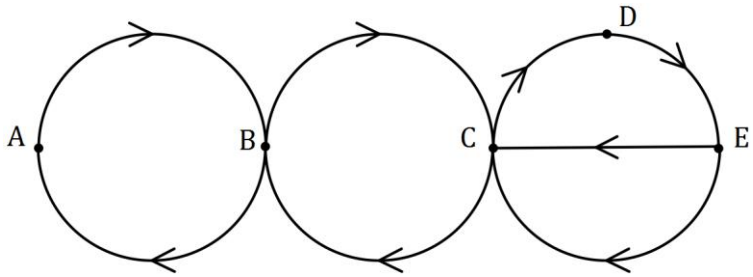
(ii) Determine the value of n if 371 is the n^{th} term of the sequence. (2 marks)

Solution
$11 + 3(n-1) = 371$ $3(n-1) = 360$ $n - 1 = 120$ $n = 121$
Specific behaviours
✓ forms equation ✓ solves equation

See next page

Question 5 **(8 marks)**

The digraph below represents a system of one-way streets that enable travel between five locations A, B, C, D and E.



(a) Complete the adjacency matrix below for the digraph. (2 marks)

	A	B	C	D	E
A	0	1	0	0	0
B	1	0	1	0	0
C	0	1	0	1	0
D	0	0	0	0	1
E	0	0	2	0	0

Solution
See table
Specific behaviours
✓ at least 3 rows correct
✓ all correct

(b) State whether a closed walk of length 5 can start from vertex B. If yes, list the vertices on the walk. If no, explain why not. (2 marks)

Solution
Yes $B\ C\ D\ E\ C\ B$
Specific behaviours
✓ states yes
✓ list of vertices

(c) The graph is semi-Hamiltonian. Clearly explain what this means. (2 marks)

Solution
An open path exists in the graph that includes every vertex once.
Specific behaviours
✓ states path visits all vertices just once
✓ states path is open

(d) List, in order, a set of vertices that must be visited to create a trail that includes every edge of the graph just once. (2 marks)

Solution
$E\ C\ B\ A\ B\ C\ D\ E\ C$
Specific behaviours
✓ starts at either E and ends at C or vice versa
✓ correct listing

See next page

Question 6 **(10 marks)**

The following table shows the scores of four people, Ali, Bo, Cee and Dip after taking four tests in economics (E), geography (G), math (M) and physics (P).

	Ali	Bo	Cee	Dip
E	12	11	10	9
C	9	13	14	11
G	13	12	11	14
P	11	9	10	13

Each of the four people are to be assigned to one of the four tests so that the total score is maximised. No-one can be assigned to more than one test.

(a) Explain why the Hungarian algorithm may be used to find the optimal assignment if each number in the table, n, is replaced by $14 - n$. (2 marks)

Solution
Hungarian algorithm is used to find the minimum assignment. To maximise we must minimise the difference from the largest score (14).
Specific behaviours
✓ states algorithm used to minimise ✓ explains reason for $14 - n$

(b) Form a new table by replacing each number in the original table, n, with $14 - n$. (1 mark)

	Ali	Bo	Cee	Dip
E	2	3	4	5
G	5	1	0	3
M	1	2	3	0
P	3	5	4	1

Solution
See table
Specific behaviours
✓ correct values

(c) Show that, by reducing **rows first** and then columns, the resulting table is as shown at the top of the next page. (2 marks)

Solution

Reduce rows:

	Ali	Bo	Cee	Dip
E	0	1	2	3
G	5	1	0	3
M	1	2	3	0
P	2	4	3	0

Reduce column:

	Ali	Bo	Cee	Dip
E	0	0	2	3
G	5	0	0	3
M	1	1	3	0
P	2	3	3	0

Specific behaviours

✓ reduces three rows 1 and 4
✓ reduces column 2

See next page

	Ali	Bo	Cee	Dip
E	0	0	2	3
G	5	0	0	3
M	1	1	3	0
P	2	3	3	0

(d) Show that the zeros in the table above can be covered with two horizontal lines and one vertical line. Hence use the Hungarian algorithm to reduce the table to a form where four lines are required to cover all zeros. (2 marks)

Solution

Subtract 1 from uncovered numbers and add 1 to those covered twice:

	Ali	Bo	Cee	Dip
E	0	0	2	4
G	5	0	0	4
M	0	0	2	0
P	1	2	2	0

See alternative shading below

Specific behaviours

✓ shading or lines
✓ reduces

(e) Determine how each of the people should be assigned to the four tests to maximise the total score, and state what this maximum score is. (3 marks)

Solution

Alternative shading

	Ali	Bo	Cee	Dip
E	0	0	2	4
G	5	0	0	4
M	0	0	2	0
P	1	2	2	0

Ali	Bo	Cee	Dip
Economics	Math	Geography	Physics

or

Ali	Bo	Cee	Dip
Math	Economics	Geography	Physics

Maximum score is $12 + 12 + 14 + 13 = 13 + 11 + 14 + 13 = 51$

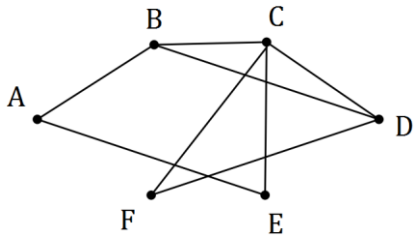
Specific behaviours

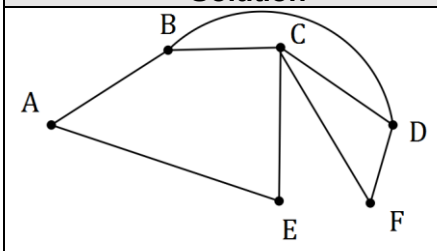
✓ shades optimal assignment
✓ lists people and tests
✓ states maximum score

See next page

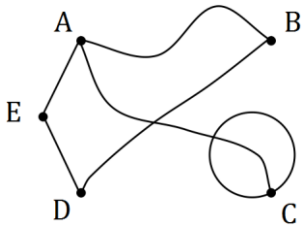
Question 7 **(8 marks)**

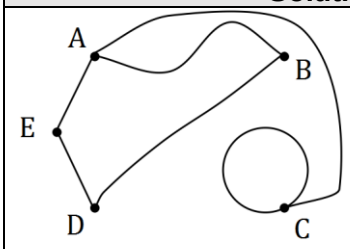
(a) Redraw the following graph to clearly show that it is planar. (2 marks)



Solution

Specific behaviours
✓ no edges that cross ✓ correct equivalent graph

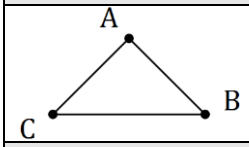
(b) Verify Euler's formula for the graph below. (3 marks)



Solution
 $V = 5, F = 3, E = 6$ $V + F - E = 2 \Rightarrow 5 + 3 - 6 = 2$
Specific behaviours
✓ redraws as planar ✓ correctly counts edges, faces and regions ✓ verifies formula

(c) Let K_n be a complete graph with n vertices.

(i) Draw the graph K_3. (1 mark)

Solution

Specific behaviours
✓ correct graph

(ii) Determine the total number of edges graph K_4 has. (1 mark)

Solution
6 edges
Specific behaviours
✓✓ correct number

(iii) State, in terms of n, the total number of edges graph K_n has. (1 mark)

Solution
$\frac{1}{2}(n \times (n-1))$ edges
Specific behaviours
✓ correct expression

End of questions

Additional working space

Question number: _____